THE RESULTANT OF A SET OF HOMOGENEOUS LINEO-LINEAR EQUATIONS.

By Thomas Muir, LL.D., F.R.S.

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1. In several of Sylvester's writings of the year 1863 there crop up references to a peculiar entity called a "double determinant." At the beginning of a paper "On a Question of Compound Arrangement,"* he says that the question arose in the course of his "successful but as yet unpublished researches into the Theory of Double Determinants"; and, having enumerated two results in connection with the said question, he appends an observation to the effect that "a double determinant means the resultant of a system of $m+n-1$ homogeneous equations, each containing mn terms, and linear in respect to each of two systems of m and n variables taken separately but of the second order in respect to the variables of these two systems taken collectively." He also states that it can be represented by an ordinary determinant of the $(m+n-1)$th order, whose elements are sums of simple determinants of the same order. Here, however, some error must have crept in, as he says quite correctly that the degree of the resultant in respect of the coefficients is

$$\frac{(m+n-1)\,!}{(m-1)\,!\,(n-1)\,!},$$

a number which implies that the order-number first mentioned should not be $m+n-1$ but

$$\frac{(m+n-2)\,!}{(m-1)\,!\,(n-1)\,!}.$$

His observation concludes with the statement that the only case previously considered is the case where $m, n=2, 2$, and that for this Cayley is responsible.†

* *Proceed. R. Soc. of London*, xii. pp. 561–563; or *Collected Math. Papers*, ii. pp. 325–326.

† Cayley's eliminant is of the 2nd order with elements of the 3rd order, in agreement with our surmise. See *Cambridge and Dub. Math. Journ.*, ix. p. 171.

2. The interest of the matter does not at all lie in the difficulty of effecting elimination, but in identifying the peculiar form of eliminant referred to by Sylvester, and in ascertaining the process devised for establishing it.

Thus, taking a set of equations illustrating the case where $m, n = 3, 2$, namely,

$$\left.\begin{array}{l} a_1\xi u + a_2\eta u + a_3\zeta u + a_4\xi v + a_5\eta v + a_6\zeta v = 0 \\ b_1\xi u + b_2\eta u + b_3\zeta u + b_4\xi v + b_5\eta v + b_6\zeta v = 0 \\ c_1\xi u + c_2\eta u + c_3\zeta u + c_4\xi v + c_5\eta v + c_6\zeta = v0 \\ d_1\xi u + d_2\eta u + d_3\zeta u + d_4\xi v + d_5\eta v + d_6\zeta v = 0 \end{array}\right\}$$

and writing each equation in the form

$$(a_1u + a_4v)\xi + (a_2u + a_5v)\eta + (a_3u + a_6v)\zeta = 0$$

we readily obtain four partial resultants like

$$\begin{vmatrix} a_1u+a_4v & a_2u+a_5v & a_3u+a_6v \\ b_1u+b_4v & b_2u+b_5v & b_3u+b_6v \\ c_1u+c_4v & c_2u+c_5v & c_3u+c_6v \end{vmatrix} = 0,$$

or

$$|a_1b_2c_3|u^3 + \left\{ |a_1b_2c_6| + |a_1b_5c_3| + |a_4b_2c_3| \right\} u^2v$$
$$+ \left\{ |a_1b_5c_6| + |a_4b_2c_6| + |a_4b_5c_3| \right\} uv^2 + |a_4b_5c_6|v^3 = 0,$$

and thence finally

$$\begin{Vmatrix} |a_1b_2c_3| & |a_1b_2c_6| - |a_1b_3c_5| + |a_2b_3c_4| & |a_1b_5c_6| - |a_2b_4c_6| + |a_3b_4c_5| & |a_4b_5c_6| \\ |a_1b_2d_3| & |a_1b_2d_6| - |a_1b_3d_5| + |a_2b_3d_4| & |a_1b_5d_6| - |a_2b_4d_6| + |a_3b_4d_5| & |a_4b_5d_6| \\ |a_1c_2d_3| & |a_1c_2d_6| - |a_1c_3d_5| + |a_2c_3d_4| & |a_1c_5d_6| - |a_2c_4d_6| + |a_3c_4d_5| & |a_4c_5d_6| \\ |b_1c_2d_3| & |b_1c_2d_6| - |b_1c_3d_5| + |b_2c_3d_4| & |b_1c_5d_6| - |b_2c_4d_6| + |b_3c_4d_5| & |b_4c_5d_6| \end{Vmatrix} = 0. \quad \text{(A)}$$

Here we have first eliminated ξ, η, ζ, and thereafter u^3, u^2v, uv^2, v^3: but we might just as well have eliminated u, v to begin with, and thereafter ξ^2, η^2, ζ^2, $\xi\eta$, $\eta\zeta$, $\zeta\xi$, the result then being

$$\begin{Vmatrix} |a_1b_4| & |a_2b_5| & |a_3b_6| & |a_2b_6| + |a_3b_5| & |a_1b_6| + |a_3b_4| & |a_1b_5| + |a_2b_4| \\ |a_1c_4| & |a_2c_5| & |a_3c_6| & |a_2c_6| + |a_3c_5| & |a_1c_6| + |a_3c_4| & |a_1c_5| + |a_2c_4| \\ |a_1d_4| & |a_2d_5| & \multicolumn{4}{c}{\dotfill} \\ |b_1c_4| & \multicolumn{5}{c}{\dotfill} \\ |b_1d_4| & \multicolumn{5}{c}{\dotfill} \\ |c_1d_4| & \multicolumn{5}{c}{\dotfill} \end{Vmatrix} = 0. \quad \text{(B)}$$

In general the compound determinant in (A) would be of the order $(m+n-1)!/m!\,(n-1)!$, and the order of the determinants forming its elements would be m: similarly in (B) the order of the compound determinant would be $(m+n-1)!/n!\,(m-1)!$ and the order of the determinants forming the elements would be n. In either case the degree of the resultant would thus be $(m+n-1)!/(m-1)!\,(n-1)!$, as we know that it ought to be.

3. Now, neither of these forms is the form which Sylvester considered satisfactory. In a paper " On a Theorem relating to Polar Umbræ,"* the nature of which may be more readily recognised by saying that it concerns a vanishing product of what Hankel afterwards called " alternate numbers," he makes the remark that " this is the fundamental theorem by aid of which I obtained the resultant of a lineo-linear system of equations in its most perfect form. It is easy to obtain two different solutions, each of them unsymmetrical in respect of the data of the question: the conversion and fusion of each of these into one and the same determinant, symmetrical in all its relations to the data, is effected instantaneously by a process derived from the above theorem."

Doubtless the two solutions which Sylvester here depreciates are those which we have just given: but we are still kept in the dark as to the solution which he considered superior.

4. Meanwhile Cayley had contributed to the *Philosophical Magazine* a theorem or two on so-called "canonic roots," which led Sylvester to publish a sequel: and in this sequel,† after referring to the contents of Cayley's paper, he says: " This is the essence of the matter communicated by Mr. Cayley: but subsequent successive generalisations of the theorem have led me on, step by step, to the discovery of a vast general theory of double determinants, that is, resultants of bipartite lineo-linear equations, constituting, I venture to predict, the dawn of a new epoch in the history of modern algebra and the science of tactic." In the course of the account which follows we see how in his investigation on " canonic roots" a special set of lineo-linear equations (m, $n=3$, 2) had turned up, why elimination was necessary, and how the idea of double determinants had arisen. Further, although the coefficients of the set of equations in question are not independent, being those of a persymmetrix matrix, we are able from an examination of the resultant obtained to guess what it would have been had the coefficients been perfectly general, like those we have used above, namely,

$$\begin{vmatrix} |a_1b_2c_3d_4| & |a_1b_3c_4d_5|-|a_1b_2c_4d_6| & |a_1b_4c_5d_6| \\ |a_1b_2c_3d_5| & |a_2b_3c_4d_5|-|a_1b_2c_5d_6| & |a_2b_4c_5d_6| \\ |a_1b_2c_3d_6| & |a_2b_3c_4d_6|-|a_1b_3c_5d_6| & |a_3b_4c_5d_6| \end{vmatrix}=0. \quad \text{(C)}$$

* *Proceed. R. Soc. London*, xii. pp. 563–565; or *Collected Math. Papers*, ii. pp. 327–328.

† *Philos. Magazine*, xxv. pp. 453–460; or *Collected Math. Papers*, ii. pp. 331–337.

The superiority of this over the two others is at once apparent. It is not only that the eliminant is of lower order: there is the further advantage that the determinants which compose its elements are all *primary* minors of the given array of coefficients.

If we use

$$(\alpha, \beta, \gamma, \dots)(m, n, r, \dots)$$

to stand for

$$\alpha m, \beta m, \gamma m, \dots, \alpha n, \beta n, \gamma n, \dots, \alpha r, \beta r, \gamma r, \dots$$

and denote a primary minor of an oblong array by the numbers of the columns which it occupies in the array, the result may be formulated thus:

The resultant of the set of equations

$$\left(\begin{array}{cccc} a_1 & a_2 & \dots & a_6 \end{array}\right)(\xi, \eta, \zeta)(u_1 v) = 0$$
$$\left|\begin{array}{cccc} b_1 & b_2 & \dots & b_6 \\ c_1 & c_2 & \dots & c_6 \\ d_1 & d_2 & \dots & d_6 \end{array}\right|$$

is

$$\left|\begin{array}{ccc} 1234 & 1345-1246 & 1456 \\ 1235 & 2345-1256 & 2456 \\ 1236 & 2346-1356 & 3456 \end{array}\right| = 0.$$

A widely general result including this—in fact, the result for the case $m, n = m, 2$—was given by me in 1905 before the conception of double determinants had been brought to my notice.*

5. So far as can be learned, Sylvester never returned to the subject. The question therefore remains as to how the result just given can be established. According to himself, he obtained it by a sort of condensation-process performed on either of the other two: but what the exact nature of this process was it is hard to guess, and practically no help is got from being told that it was dependent on his theorem about polar umbræ.

6. It is therefore all the more interesting to know that it can be obtained quite independently of the two others and as easily as either. For, writing the four given equations in the form

$$a_1 u\xi + a_4 v\xi + (a_2 u + a_5 v)\eta + (a_3 u + a_6 v)\zeta = 0,$$

and eliminating $u\xi$, $v\xi$, η, ζ, we have

$$\left|\begin{array}{cccc} a_1 & a_4 & a_2 u + a_5 v & a_3 u + a_6 v \\ b_1 & b_4 & b_2 u + b_5 v & b_3 u + b_6 v \\ c_1 & c_4 & c_2 u + c_5 v & c_3 u + c_6 v \\ d_1 & d_4 & d_2 u + d_5 v & d_3 u + d_6 v \end{array}\right| = 0,$$

* *Transac. R. Soc. Edinburgh*, xlv. pp. 1–7; and *Messenger of Math.*, xxxv. pp. 118–121.

whence

$$|1423|u^2 + \left\{ |1426| + |1453| \right\} uv + |1456| v^2 = 0.$$

Similarly there is obtained

$$|1253|u^2 + \left\{ |1256| + |4253| \right\} uv + |4256| v^2 = 0,$$

by eliminating ξ, $u\eta$, $v\eta$, ζ: and

$$|1236|u^2 + \left\{ |1536| + |4236| \right\} uv + |4536| v^2 = 0,$$

by eliminating ξ, η, $u\zeta$, $v\zeta$. Elimination of u^2, uv, v^2 from these three partial resultants is then all that is required.

7. Further, this process of ours is of perfectly general application. The $m+n-1$ given equations being

$$\begin{matrix} (a_1 & a_2 & \cdots & a_{mn} \end{matrix} \between (x_1, x_2, \ldots, x_m)(y_1, y_2, \ldots, y_n) = 0$$
$$\left| \begin{matrix} b_1 & b_2 & \cdots & b_{mn} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \multicolumn{4}{c}{(m+n-1 \text{ rows})} \end{matrix} \right|$$

we eliminate the x's in such a way as to obtain a set of

$$(m+n-2)!/(m-1)!\,(n-1)!$$

equations linear and homogeneous in the compound variables

$$y_1^{m-1},\quad y_1^{m-2}y_2,\quad \ldots,\quad y_n^{m-1},$$

and then at one stroke eliminate these also. Or, we may begin by eliminating the y's in such a way as to obtain a set of equations linear and homogeneous in the compound variables

$$x_1^{n-1},\quad x_1^{n-2}x_2,\quad \ldots,\quad x_m^{n-1},$$

and then eliminate the x's.

The number of compound variables in the one set is the same as in the other, because the number of combinations of m things taken $n-1$ together with repetitions allowed is the same as the number of the similar combinations of n things taken $m-1$ together, namely,

$$(m+n-2)!/(m-1)!\,(n-1)!$$

Further, it is most important to notice that whether we begin with the elimination of the x's or the elimination of the y's there is no real difference in the result, the rows of the one eliminant being the columns of the other.

8. The mode of effecting the preliminary set of eliminations is well worth a little attention. To make it clear, let us take the case where $m, n = 3, 3$, namely, the case of the elimination of $\xi, \eta, \zeta, u, v, w$ from the five equations

$$\left.\begin{array}{c} a_1\xi u + a_2\eta u + a_3\zeta u + a_4\xi v + \ldots + a_9\zeta w = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ e_1\xi u + e_2\eta u + e_3\zeta u + e_4\xi v + \ldots + e_9\zeta w = 0 \end{array}\right\}$$

Arranging the equations in the form

$$a_1\xi u + a_2\eta u + a_3\zeta u + (a_4\xi + \ldots)v + (a_7\xi + \ldots)w = 0,$$

we eliminate $\xi u, \eta u, \zeta u, v, w$, and obtain

$$\begin{vmatrix} a_1 & a_2 & a_3 & a_4\xi + a_5\eta + a_6\zeta & a_7\xi + a_8\eta + a_9\zeta \\ \cdots & \cdots & \cdots & \cdots\cdots & \cdots\cdots \\ e_1 & e_2 & e_3 & e_4\xi + e_5\eta + e_6\zeta & e_7\xi + e_8\eta + e_9\zeta \end{vmatrix} = 0,$$

whence there results

$$\begin{aligned} &12347\xi^2 + 12348\xi\eta + 12349\xi\zeta \\ &+ 12357\eta\xi + 12358\eta^2 + 12359\eta\zeta \\ &+ 12367\zeta\xi + 12368\xi\eta + 12369\zeta^2 = 0. \end{aligned}$$

By the elimination of $u, \xi v, \eta v, \zeta v, w$, we obtain a similar equation, and by the elimination of $u, v, \xi w, \eta w, \zeta w$ a third. We have thus three equations in the six compound variables $\xi^2, \eta^2, \zeta^2, \xi\eta, \eta\zeta, \zeta\xi$, and have therefore to seek for other three. It is the obtaining of the latter three that is most interesting.

Putting the given equations in the form

$$(a_1\xi + a_2\eta)u + a_3\zeta u + a_4\xi v + (a_5\eta + a_6\zeta)v + (a_7\xi + a_8\eta + a_9\zeta)w = 0,$$

eliminating $u, \zeta u, \xi v, v, w$, and proceeding as before, we reach a cubic in ξ, η, ζ, every term of it involving η except two, namely,

$$13467\xi^2\zeta,\ 13469\xi\zeta^2.$$

By elimination of $\xi u, u, v, \zeta v, w$ there is reached in the same way a cubic of the same kind, alike even in the particular that all its terms involve η except

$$13467\xi^2\zeta,\ 13469\xi\zeta^2.$$

The performance of subtraction thus enables us to divide by η and to arrive at an equation containing only the variables ζ^2, η^2, ζ^2, $\xi\eta$, $\eta\zeta$, $\zeta\xi$, namely, the equation

$$\begin{aligned}(-12467+13457)\xi^2+(-12568+23458)\eta^2+(-13569+23469)\zeta^2 \\ +(-12468+13458-12567+23457)\xi\eta \\ +(-12569+23459-13568+23468)\eta\zeta \\ +(-12469+13459-13567+23467)\zeta\xi = 0.\end{aligned}$$

Similarly it is found that

$$\begin{aligned}(-14597+14678)\xi^2+(-24589+25678)\eta^2+(-34689+35679)\zeta^2 \\ +(-24579+24678-14589+15678)\xi\eta \\ +(-34589+35678-24689+25679)\eta\zeta \\ +(-34579+34678-14689+15679)\zeta\xi = 0,\end{aligned}$$

and that

$$\begin{aligned}(-13478+12479)\xi^2+(-23578+12589)\eta^2+(-23679+13689)\zeta^2 \\ +(-13578+12579-23478+12489)\xi\eta \\ +(-23678+12689-23579+13589)\eta\zeta \\ +(-13678+12679-23479+13489)\zeta\xi = 0.\end{aligned}$$

The final eliminant is thus a determinant of the 6th order, each of whose elements is either a five-line determinant or a sum of five-line determinants.

9. Almost all this work, however, is unnecessary, in view of the fact that the original set of equations is invariant to the simultaneous circular substitutions

$$\xi,\ \eta,\ \zeta=\eta,\ \zeta,\ \xi;\quad \begin{matrix}1,\ 2,\ 3=2,\ 3,\ 1; \\ 4,\ 5,\ 6=5,\ 6,\ 4; \\ 7,\ 8,\ 9=8,\ 9,\ 7;\end{matrix} \qquad (S_1)$$

and also to the simultaneous circular substitutions

$$u,\ v,\ w=v,\ w,\ u;\quad \begin{matrix}1,\ 4,\ 7=4,\ 7,\ 1 \\ 2,\ 5,\ 8=5,\ 8,\ 2 \\ 3,\ 6,\ 9=6,\ 9,\ 3.\end{matrix} \qquad (S_2)$$

Making use of this invariance, we can construct the eliminant with ease as soon as two elements of the first row are known.

Thus, the first element of all being 12347, substitution S_1 gives 23158, 31269 for the next two elements of the first row: and the fourth element being the sum of the first two elements with columns 7 and 8 interchanged, the fifth element will be the sum of the second and third with columns 8 and 9 interchanged, and the sixth will be the sum of the third and first with columns 9 and 7 interchanged. Then, having thus got the first element of every column, substitution S_2 suffices to give all the other elements ; for example, the first element of the fourth column being

$$12348+12357,$$

the next two are

$$45672+45681 \quad \text{and} \quad 78915+78924 :$$

and the fourth being the sum of the first and second with columns 3 and 6 interchanged, namely,

$$12648+12657+45372+45381,$$

it follows that the remaining two are

$$\begin{gathered} 45972+45981+78615+78624, \\ 78315+78324+12948+12957. \end{gathered}$$

10. Further, since each of the given equations may be written in bipartite notation,

$$\begin{array}{ccc|c} \xi & \eta & \zeta & \\ \hline a_1 & a_2 & a_3 & u \\ a_4 & a_5 & a_6 & v \\ a_7 & a_8 & a_9 & w \end{array} = 0, \quad \begin{array}{ccc|c} \xi & \eta & \zeta & \\ \hline b_1 & b_2 & b_3 & u \\ b_4 & b_5 & b_6 & v \\ b_7 & b_8 & b_9 & w \end{array} = 0, \quad \ldots ;$$

and as each row of the eliminant consists of the coefficients of an equation in ξ^2, η^2, ζ^2, $\xi\eta$, $\eta\zeta$, $\zeta\xi$, and each column consists of the coefficients of an equation in u^2, v^2, w^2, uv, vw, wu, it would not be difficult to devise a rule for writing down any element of the eliminant quite independently of the others. Thus, the element in the (3,2)th place being the coefficient of η^2 in the one set of equations and of w^2 in the other set must have its column-numbers taken from the mutilated bipartite

$$\begin{array}{ccc|c} \xi & \eta & \zeta & \\ \hline . & 2 & . & u \\ . & 5 & . & v \\ 7 & 8 & 9 & w, \end{array}$$

and, as a matter of fact, it is 25789.